

Correlation Risk

Rischio di Correlazione

Cherubini Umberto

University of Bologna, MatematES,

Viale Filopanti 5 40126 Bologna, e-mail:Umberto.Cherubini@unibo.it

Riassunto: In questo lavoro vengono passati in rassegna i principali modelli di analisi del rischio di correlazione. Viene discussa la rilevanza di tale fonte di rischio sia per le decisioni di investimento che per le attività di valutazione delle attività finanziarie e misurazione del rischio. Tra i diversi modelli disponibili per rappresentare il rischio di correlazione, viene approfondito quello delle funzioni di copula. A tale proposito viene presentato un modello che impiega le funzioni di copula per rappresentare il rischio di correlazione sia in una prospettiva di serie storica che di cross-section. Viene mostrato l'utilizzo del modello su un titolo multivariato con barriera (Altiplano) di largo utilizzo in prodotti finanziari e assicurativi strutturati. In conclusione vengono enunciati i problemi aperti nei modelli di correlazione e i possibili sviluppi di ricerca futuri.

Keywords: Copula functions, Correlation, Risk Management, Option Pricing

1. Introduction

Correlation trading denotes the trading activity aimed at exploiting changes in correlation or more generally in the dependence structure of assets or risk factors. Likewise, correlation risk is defined as the exposure to losses triggered by changes in correlation. This contribution is a review paper of the frontier issues in correlation risk. We address three tasks: the first is to review the financial economics and statistics of correlation; the second is to present a new model merging serial and cross-section correlation; the third is to point out the keywords of research in this field for the near future. Some of the questions addressed are still open problems, at least from the point of view of this author. The aim is to extend these issues to the audience of statisticians in the hope of some “intellectual arbitrage”, that is the event that some of these issues have already found answers in other fields of applied statistics.

The paper is organized as follows. Section 2 briefly illustrates how correlation risk affects the main activities in finance, namely asset management, pricing and risk management. Section 3 provides a bird eye review of methods to model correlation and dependence in general, with particular focus on the method of copula functions. Section 4 presents a new model of joint asset prices dynamics, in which copulas are applied to represent both serial dynamics and cross-section dependence: application of the model to a multivariate barrier digital product (known as Altiplano) illustrates the flexibility of the approach. Section 5 presents three keywords for future research: asymmetry, non-exchangeability and Lévy copulas.

2. Correlation risk in finance.

Correlation risk is nowadays a major feature across all the main activities in finance, namely: i) asset management, ii) the pricing of financial products; iii) risk management.

Asset Management. Consider a standard static portfolio allocation problem. The aim is to maximize expected utility of wealth W at some final date T using a set of risky assets, S_i , $i = 1, 2, \dots, m$.

$$\max_{w_1, w_2, \dots, w_m} E_P \left[U \left(R_0 + \sum_{i=1}^m w_i (R_i - R_0) \right) \right] \quad (1)$$

where $R_i = \log(S_i(T)/S_i(0))$ are the log returns of the risky assets and R_0 is the risk-free rate. The asset allocation problem is completely described by two functions: i) the utility function U , assumed strictly increasing and concave; ii) the joint distribution function of the returns P . We could argue in depth about both of them. Obviously, our focus here is on the joint distribution function. In the standard textbook problem, returns are assumed to be jointly normally distributed, which leads to re-writing the expected utility in terms of a mean-variance problem. Nowadays, reality has moved miles far away from this basic problem, mainly for two reasons: first, investments are no longer restricted to linear products, such as stocks and bonds, but involve options and complex derivatives; second, the assumption that distribution of the returns is gaussian is clearly rejected by the data. As a result, the expected utility problem should take into account three different dimensions of risk: i) directional movements of the market; ii) changes in volatility of the assets; iii) changes in their correlation. More important, there is also clear evidence that changes in both volatility and correlation are themselves correlated with swings in the market. Typically, both volatility and correlation increase when the market is heading downward (which is called *leverage effect*).

Pricing. The need to account for these new dimension of risk has led to the diffusion of derivative products to allow to hedge and speculate on both changes of volatility and changes in correlation. From the point of view of the pricer, the approach is taken from the first order conditions of the investment problem. In fact, we may easily re-write them as

$$E_P \left[U' \left(R_0 + \sum_{i=1}^m w_i (R_i - R_0) \right) (R_i - R_0) \right] = 0 = E_Q [(R_i - R_0)] \quad (2)$$

where the new probability measure Q is defined after the Radon-Nikodim derivative

$$\frac{dQ}{dP} \equiv \frac{U' \left(R_0 + \sum_{i=1}^m w_i (R_i - R_0) \right)}{E_P \left[U' \left(R_0 + \sum_{i=1}^m w_i (R_i - R_0) \right) \right]} \quad (3)$$

Pricers tackle the problem of evaluating financial products using measure Q , which is called the *risk-neutral* measure (because all the risky assets are expected to yield the

same return as the risk-free asset), or the *equivalent martingale measure* (EMM, because Q is a measure equivalent to P with the property that prices expressed using the risk-free asset as numeraire are martingale). An open issue is whether and under which circumstances volatility and correlation of the original measure P are preserved under this change of measure. If this is not the case we say that volatility and correlation risks are priced in the market (that is, a risk premium is required for facing these risks). Under this new measure, the pricers face problems which are similar to those of the asset manager, that is evaluating the sensitivity of financial products to changes in the direction of the market (long/short the asset), volatility (long/short volatility) and correlation (long/short correlation).

They face a problem more, though, that is going to be very relevant for our discussion: they must ensure that prices of multivariate products should be consistent with prices of uni-variate products. Such consistency is part of the so-called *arbitrage-free* approach to pricing which may be proved to lead to the same martingale requirement as that presented above. In the jargon of statisticians this consistency could be referred to as *compatibility*: the risk-neutral joint distribution Q had to be compatible with the marginal distributions Q_i . We will see below that the compatibility problem in statistics is much more complex than this. For the time being, we use two real world examples, one from the equity market and the other from the credit market, to stress the point.

Example 1. Altiplano Note. These are so called *digital* products, that is paying a fixed sum if some event takes place at a given future date T . Typically the event is defined on a set of stocks or market indexes and the product pays the fixed sum if all of them are above some given level, typically specified as a percentage of the initial level. The price of this product is of course the joint *risk-neutral* probability that all the assets are above the specified level at time T : $Q(S_1(T) > K_1, S_2(T) > K_2, \dots, S_m(T) > K_m)$ where K_i are the levels. Consider now that we can actually estimate the marginal distributions from the option markets (as explained in the famous Breeden and Litzenberger, 1978 paper). So, we can price $Q_i(S_i(T) > K_i)$. As a result, the only reason why one invests in the multivariate digital product above instead of on a set of uni-variate ones is to exploit changes in correlation among the assets. To put it in other terms, the multivariate product can increase value even if the prices of all the uni-variate products remain unchanged, and this may occur if the correlation increases.

Example 2. Credit index tranches. Today it is possible to invest in portfolios of credit derivatives. In non technical terms, we can buy and sell insurance ("protection" is the market term) on the first $x\%$ losses on defaults of a set of obligors (called "names"). This product is called 0 - $x\%$ *equity tranche* of a portfolio of credit losses. Standard products in the market include 125 "names", and have equity tranches 0 - 3% , 0 - 6% and so on. For the sake of simplicity assume 100 names and a 0 - 1% equity tranche, and assume that in case of default each loss is equal to 1. So, this tranche pays 1 the first time a default occurs (it is also called a *first-to-default* insurance). Again, we can easily recover the probability of default from other products, namely the so called *credit default swap* (CDS) market. So, we can price the protection for every single name. The price of the first to default must then be compatible with such prices. In fact, with respect to such prices, the multivariate product only allows to invest on correlation. Again, the equity tranche can increase in value even though the values of single-

insurance (CDS) for all the names remain constant, provided that correlation among default of the names increase.

Risk Management. The risk manager faces the problem of measuring the exposure of the position to different risk factors. In the standard practice he transforms the financial positions in the different assets and markets into a set of *exposures* (*buckets*, in jargon) to a set of risk factors (*mapping* process). The problem is then to estimate the joint distribution of losses $L_1, L_2, L_3, \dots, L_k$. on these exposures and define a *risk measure* on this distribution. Typical measures are *Value-at-Risk* (VaR) and *Expected Shortfall* (ES) defined as

$$VaR_i \equiv H_i^{-1}(\alpha) \quad ES_i \equiv E(L_i | L_i \geq VaR_i) \quad (3)$$

where $H_i(\cdot)$ is the marginal probability distribution of loss L_i . The risk measure of the overall portfolio will analogously be

$$VaR_i \equiv H^{-1}(\alpha) \quad ES \equiv E\left(\sum_{i=1}^k L_i | L_i \geq VaR\right) \quad (4)$$

where $H(\cdot)$ is now the probability distribution of the sum of losses. The aggregate measure would account for correlation among the losses, but there is a subtle point that is to be noticed, and makes this practice obsolete with respect to correlation products. Consider products like the Altiplano note above. A first point is that this product is non linear, so that its value may change even though market prices do not move but their volatility do. This point can be handled by including a *bucket* of volatility exposures for every risk factor. But there is a crucial point that gets lost in the procedure. It is the fact that the value of the product can change even if neither the market prices or their volatility change, but because of a change in correlation. This exposure to correlation among the assets included in the specific product has gotten lost in the *mapping* procedure. Correlation risk then induce risk manager to measure this dimension of risk on a product by product basis, using either historical simulation or stress testing techniques.

3. Modelling Correlation

In a very general and stylised taxonomy correlation models can be classified according to whether they model the correlation parameter in continuous time (*instantaneous correlation*) or in discrete time (*terminal correlation*).

Continuous time A possible choice to model the dynamics of correlation in continuous time is to specify a matrix representation of a stochastic differential equation. This is actually done in the so called Wishart models (Gourieroux and Sufana, 2004 and De Fonseca et al., 2004): the dynamics of the whole variance-covariance matrix is specified in continuous time as a multi-variate square root process. An alternative route is to model the instantaneous correlation parameter directly as a new risk factor. The natural choice is that of a Jacobi process, which by its very nature enables to restrict the dynamics in the $[-1, 1]$ interval, and this is the model used in Driessen et al. (2005).

Copula functions. Among the models used for the specification of correlation in discrete time, copula functions have experienced a large success in financial applications (Cherubini et al, 2004). The reason is that this approach enables to separate specification of the marginal distributions and the dependence structure: correlation risk can then be singled out and priced while keeping marginal distributions “compatible” with uni-variate prices. Compatibility is actually hearth of the matter here. Technically, the idea is built upon the principle of probability integral transformation. For example, given marginal probability Q_i^T of asset $S_i(T)$ we know that $u_i = Q_i^T(S_i(T))$ is uniformly distributed in the interval $[0,1]$. If the distributions are strictly monotone we may write the joint distribution as

$$Q(S_1 \leq K_1, S_2 \leq K_2, \dots, S_m \leq K_m) = Q(Q_1^{-1}(u_1), Q_2^{-1}(u_2), \dots, Q_m^{-1}(u_m)) \quad (5)$$

where we dropped reference to T for the ease of notation. Every joint distribution can be written as a function of the marginals.

$$Q(S_1 \leq K_1, S_2 \leq K_2, \dots, S_m \leq K_m) = C(u_1, u_2, \dots, u_m) \quad (6)$$

Conditions which must be satisfied by this function are spelled out in Sklar’s theorem. They are basically three. First, the function must map the unit hypercube to the unit interval. Second, the function must be *grounded*, meaning that $C(u_1, u_2, \dots, 0, \dots, u_m) = 0$; furthermore, it must be $C(1, 1, \dots, u_i, \dots, 1) = u_i$ for all $i = 1, 2, \dots, m$. Finally, the function must be m -increasing: in other words, given two m -dimension vectors $\mathbf{u} < \mathbf{v}$, the volume between the two vectors cannot have negative measure. For clarity, consider the simplest case $m = 2$. In this case the volume is defined as

$$V_C[\mathbf{u}, \mathbf{v}] = C(v_1, v_2) - C(u_1, v_2) - C(u_2, v_1) + C(u_1, u_2) \geq 0 \quad (7)$$

Copula functions grant maximum flexibility to model dependence structure among variables, separately from marginal distribution specifications. For this reason they are naturally linked to non parametric measures of association, as well as to measures of tail dependence. For a complete coverage of the mathematics and statistics of copula functions the reader is referred to Nelsen (2006) and Joe (1997). Here we will just focus on some of the properties that are relevant for applications to finance and to our particular discussion here. An important feature, that is paramount in pricing applications, is the relationship between copula functions representing the joint distribution of a set of events and those representing the joint distribution of the complement sets. A *survival copula* is defined as

$$Q(S_1 > K_1, S_2 > K_2, \dots, S_m > K_m) = \bar{C}(1 - u_1, 1 - u_2, \dots, 1 - u_m) \quad (5)$$

Survival copulas are naturally linked to copulas by a straightforward relationship. A simple way of thinking of this relationship is to use the concept of volume defined above. If we denote $\mathbf{e}$ the m -dimension unit vector, the survival copula can be represented as the volume of the original copula from any vector $\mathbf{u}$ to the vector $\mathbf{e}$. That is

$$\bar{C}(1-u_1, 1-u_2, \dots, 1-u_m) = V_C[\mathbf{u}, \mathbf{e}] \quad (6)$$

For a simple check, consider the bi-variate case, and remember $C(u_i, 1) = u_i$.

$$V_C[\mathbf{u}, \mathbf{e}] = C(1, 1) - C(u_1, 1) - C(u_2, 1) + C(u_1, u_2) = 1 - u_1 - u_2 + C(u_1, u_2) \quad (7)$$

The relevance of this relationship in finance is clear because it enforces no-arbitrage relationships between products paying out if a set of events take place and products paying out if no one of them take place. Going back to the Altiplano note above we may provide a straightforward check of this principle.

Example 3. Altiplano Note. Assume an Altiplano Note like that in example 1, with the only difference that the fixed sum is paid if all the assets S_i are below (instead of above) some predefined levels K_i . Clearly, the value of the product will be $Q(S_1(T) \leq K_1, S_2(T) \leq K_2, \dots, S_m(T) \leq K_m)$. Given the marginal distributions, the dependence structure of this product, which could be called *put*, or *bearish*, Altiplano should be represented by a copula, while the price of the *call* or *bullish* Altiplano in example 1 should be computed using the survival copula. It can be proved that if this is not the case, one could exploit arbitrage profits (see Cherubini et al, 2004, ch. 8).

Dynamic Conditional Correlation. As part of the Garch literature, which is very popular in finance, particularly in financial econometrics, the DCC model proposed by Engle (2002) is the most relevant challenge to the copula function approach. The reason is that this model enables to break down the specification of the multivariate dynamics of m assets in the separate specification of Garch models for uni-variate dynamics and a dynamic model for the correlation matrix. It is a very important feature that maximum likelihood estimation can be carried out in two stages just like in the IFM (*inference from the margins*) method used in the copula literature. The structure of dependence remains of course less flexible than in copula models, and the conditionally gaussian structure of dependence makes difficult to represent features such as correlation asymmetry and tail dependence in the model. Modifying the dependence structure to incorporate these features naturally leads to specifying conditional dependence using copulas. This is the *conditional copula* approach proposed by Patton (2006): marginal distributions are estimated using Garch models, while the conditional dependence is represented by a copula function, so that shocks and news affect both the marginal distributions and the dependence measures at the same time.

Implied correlation. It is peculiar to financial statistics that information can be recovered both from the history of prices of assets and from derivative contracts written on the same assets. We distinguish then between *historical information* and *implied information*. The standard example is *implied volatility*, which can be recovered from option prices. Other examples are *implied probability*, recovered from option spreads as shown in Breeden and Litzenberger (1978). It is not surprising that development of correlation trading has led to the concept of *implied correlation*: the concept is nowadays common knowledge in the credit index tranche market (see example 2), but the same concept is about to be extended to equity markets as well.

4. A Copula Based Model of Market Dynamics

So far we have described correlation in a cross-section setting. And indeed our main discussion has been devoted to contingent claims whose pay-offs were not path-dependent. In more involved cases, in which time paths of the assets are also relevant, a new dimension of correlation arises, that is time series correlation. Here we describe a general setting that exploits the flexibility of copulas to build a dynamic discrete time model of a set of assets. The idea follows Cherubini and Romagnoli (2007) and is to use copulas to model both serial and cross-section dependence in the market. The model will be then applied to the evaluation and analysis of a complex product encompassing both dimensions of correlation.

We assume a set of $\{S_1, S_2, \dots, S_m\}$ and a set of $\{t_0, t_1, t_2, \dots, t_n\}$ dates. We assume a filtered probability space generated by the prices and satisfying the usual conditions. The basic assumptions of the model are

Assumption 1. The conditional distribution of the price $S_i(t_j)$ conditional on the set of information available at time t_0 is P_i^j or Q_i^j in the risk-neutral setting.

Assumption 2. Each asset is generated by a first order Markov process. Conditional dependence between the price of asset S_i from time t_{j-1} to time t_j is represented by the copula function $T_i^{j-1,j}(u, v)$

Assumption 3. Cross-section dependence among the assets S_i at time t_j is $C^j(u_1, u_2, \dots, u_m)$.

Assumption 4. The copula function $C(Q_1^1, Q_1^2, \dots, Q_1^j, \dots, Q_m^1, Q_m^2, \dots, Q_m^j)$ admits the hierarchical representation $C(G_1^j(Q_1^1, Q_1^2, \dots, Q_1^j), \dots, G_m^j(Q_m^1, Q_m^2, \dots, Q_m^j))$ where $G_i^j(u_i^1, u_i^2, \dots, u_i^j)$, $i = 1, 2, \dots, m$ are copula functions.

Based on these assumptions we are now in a position to recover some results on both time series and cross-section compatibility.

4.1. Time Series Compatibility

We first elaborate on the dynamic structure of each asset S_i . Notice that defining the time series joint distribution of the price of the asset amounts to defining the distribution of running maximum and minimum.

Definition 1. The probability distribution of the running maximum of the price of asset S_i is defined as

$$\begin{aligned} Q_i(\max_k S_i(t_k) \leq K; k = 1, 2, \dots, j) &\equiv \\ &\equiv Q_i(S_i(t_1) \leq K, S_i(t_2) \leq K, \dots, S_i(t_j) \leq K) = G_i^j(Q_i^1, Q_i^2, \dots, Q_i^j) \end{aligned} \quad (8)$$

Definition 2. The survival probability distribution of the running minimum of the price of asset S_i is defined as

$$Q_i \left(\min_k S_i(t_k) > K; k = 1, 2, \dots, j \right) \equiv \\ \equiv Q_i \left(S_i(t_1) > K, S_i(t_2) > K, \dots, S_i(t_j) > K \right) = \bar{G}_i^j \left(1 - Q_i^1, 1 - Q_i^2, \dots, 1 - Q_i^j \right) \quad (9)$$

where $\bar{G}_i^j(1 - u_i^1, 1 - u_i^2, \dots, 1 - u_i^j)$ is the survival copula of $G_i^j(u_i^1, u_i^2, \dots, u_i^j)$,

In order to specify the model we exploit the Markov property of the price. The problem was studied in Darsow et al. (1992). The representation is based on the following operator

Definition 3. Assume two bivariate copulas $A(u, t)$ and $B(t, v)$ the product operator $*$ is defined as

$$A * B(u, v) \equiv \int_0^1 \frac{\partial A(u, t)}{\partial t} \frac{\partial B(t, v)}{\partial t} dt \quad (10)$$

Definition 4. Assume an m -dimensional copula A and an n -dimensional copula B the star- $\star$ operator is defined as

$$A \star B(u_1, u_2, \dots, u_{m+n-1}) \equiv \int_0^{u_m} \frac{\partial A(u_1, u_2, \dots, u_{m-1}, t)}{\partial t} \frac{\partial B(t, u_{m+1}, u_{m+2}, \dots, u_{m+n-1})}{\partial t} dt \quad (11)$$

We can now report the main theorem we are going to use in the following

Theorem. 1. The stochastic process S_i is a stationary Markov chain of first order if and only if

$$G_i^j(u_i^1, u_i^2, \dots, u_i^j) = T_i^{1,2}(u_i^1, u_i^2) \star T_i^{2,3}(u_i^1, u_i^2) \dots \star T_i^{j-1,j}(u_i^1, u_i^2) \quad (12)$$

The theorem was proved by Darsow et al. (1992), which gave necessary and sufficient conditions for a time series process based on bivariate copulas to be first order Markov (Ibragimov 2005 provided extensions to k order Markov processes).

The above result provides immediate specification of the running maximum of our time series for every time horizon. The distribution of the running minimum is then obtained by computing the survival copula. This result can further be merged with the recursive algorithm proposed in Cherubini and Romagnoli (2007) to yield a very useful procedure for the analysis of market dynamics. The recursion may be recovered from *Lemma 1*. Assume $C(u, v, z)$, $C_1(v, z) = \min(v, z)$, and $C^*(u, v)$. Then it must be $C^*(u, z)$.

Proof is trivial and relies on the fact that v and z are perfectly dependent, and may be written as an increasing function of each other. The property that copulas are invariant to increasing transformations complete the proof.

We have everything we need to prove

Theorem. 2. If the price process S_i follows a first order Markov process, its running maximum G_i^j can be computed by a recursive procedure based on G_i^{j-1} and Q_i^j .

$$G_i^j = T_i^{j-1,j}(G_i^{j-1}, Q_i^j) \quad (13)$$

Proof follows observing that the first order Markov process allows to write the probability of the running maximum at time t_j as a function of the probability running maximum at time t_{j-1} and the marginal probability of the level at time t_j . Then, we observe that $\text{Prob}(G_i^{j-1}(K), Q_i^{j-1}) = \min(G_i^{j-1}(K), Q_i^{j-1}(K))$, because the event that t_{j-1} running maximum be lower than K is included in the event that $S_i(t_{j-1})$ be lower than K . Lemma 1 and assumption 2 then deliver the result.

Based on this result, it is possible to build a “bootstrapping” procedure to construct the running maxima distribution from the conditional probabilities of prices and their Markov dependence structure ⁽¹⁾. Vice versa, if markets were so developed to provide reliable quotes for barrier options, transition probabilities could be backed out from the data. Notice that the corresponding procedure for running maxima is not straightforward, because at every step t_j it should preserve the relationship between the copula function representing the running maximum and the survival copula representing the running minimum. Cherubini and Romagnoli (2007) provide necessary restrictions to preserve such relationship, however they are of little practical help, because they require at any step t_j to compute a volume in dimension $j - 1$. An efficient algorithm to compute survival copulas is however provided, we refer to the paper for details.

The result above is very useful to establish arbitrage relationships between barrier options. Particularly, barriers may be directly computed in their own discrete time setting, rather than in continuous time and then approximated, as it is usually done in financial mathematics. The relationship between a plain vanilla product and a barrier product is in fact nothing but the relationship between the probability distribution of the price of an asset and that of the corresponding running maximum (or minimum). An example is reported in Figure 1 in which we derive such relationship for two different examples of copulas (gaussian and Clayton).

¹ Here we borrow the term “bootstrap” from the financial jargon where it is used to describe how to recover the term structure of yields from swap rates, default probabilities from CDS premia, caplets volatilities from cap volatilities, and the like. There is no connection to the bootstrap concept in statistics for data simulation.

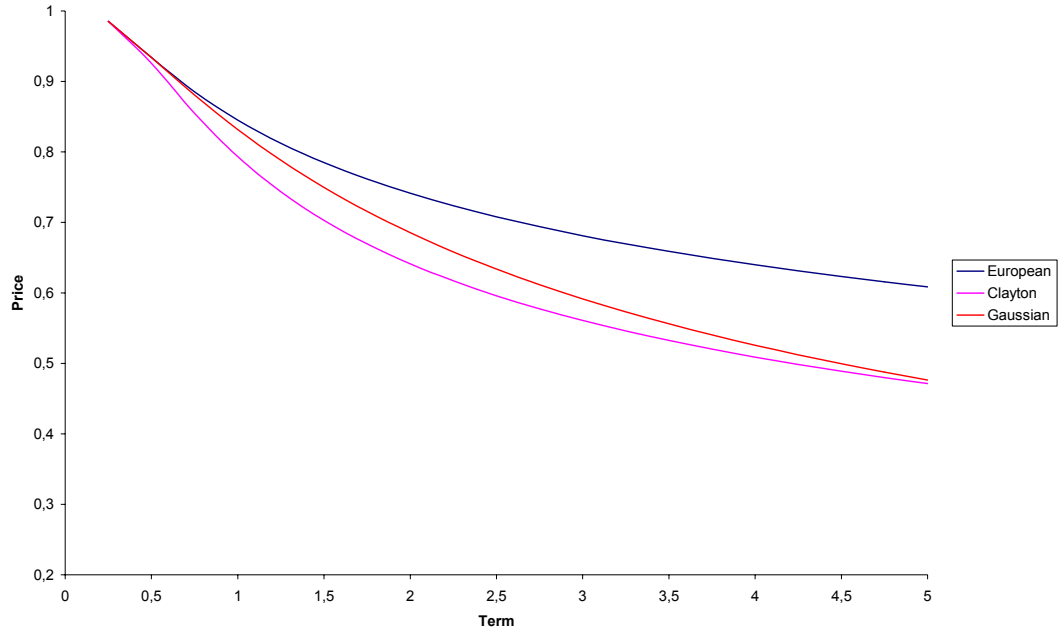


Figure 1. Prices of European and barrier digital products. Quarterly monitoring.

4.2. Cross-section Compatibility

Cross section compatibilities were imposed from the beginning under Assumption 3. We can however easily establish relationships between the time-series and the cross-section dependence structure. In other words, we recover dependence relationships among running maxima, or among running minima. The key to the result is again lemma 1, by which it is immediate to prove.

Theorem 3. The cross-section dependence structure of running maxima and minima is given by

$$C^j(G_1^j, G_2^j, \dots, G_m^j) \quad \bar{C}^j(1 - G_1^j, 1 - G_2^j, \dots, 1 - G_m^j) \quad (14)$$

As before, the proof relies on the fact that the event of the running maximum of asset S_i at time t_j being lower than a given level is contained in the event of $S_i(t_j)$ being lower than the same level: that is, if the first event occurs, the other must occur as well, so that the two events are perfectly correlated. The same argument can then be applied to the running minimum at time t_j and the event of $S_i(t_j)$ being higher than the same level.

We may now analyze both the perspectives of correlation in a special kind of financial derivative, that is very often embedded in structured notes and life insurance products.

Example 3. Barrier Altiplano Note. Differently from the multivariate digital product in example 1, which pays a fixed sum if all of the assets S_i in a basket are above a given threshold **at** some future time t_n , in a barrier Altiplano the payment is done if they remain above the threshold **by** that date. Monitoring is typically done in discrete time, mainly on a monthly basis. Of course, this product encompasses both time series and cross-section dependence. The value of the product may be negatively affected by

changes in correlation in both senses. Furthermore, the argument that the probability of the running maxima (or minima) of the prices are lower than the probability of the price level establishes that barrier Altiplanos must be cheaper than European Altiplanos. Figure 2 reports a comparison of the two products.

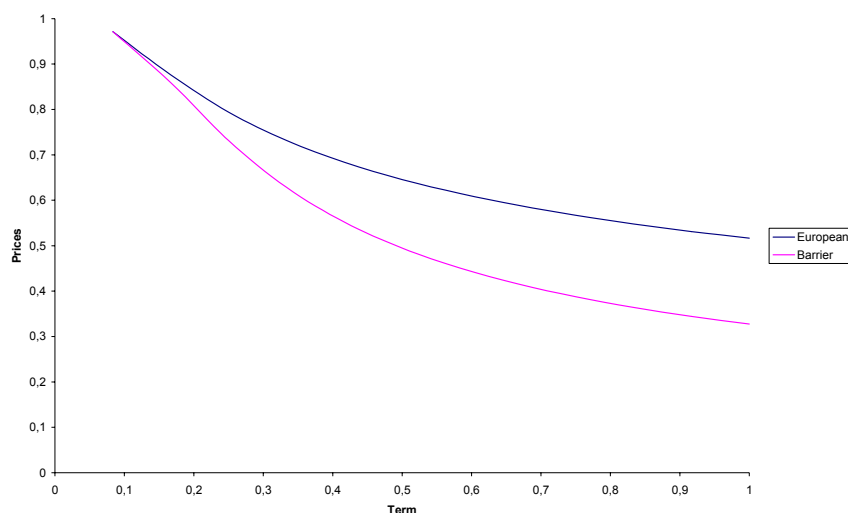


Figure 2. *Prices of European and barrier Altiplanos.*

5. Frontier issues

We conclude our analysis pointing out some frontier questions in correlation modelling, which could make our general model in section 4 much richer.

Dependence asymmetry. Several examples of asymmetry have been studied in finance, with particular applications to asset management and risk management. Mostly, attention have been focussed on the increase correlation in periods when market falls (Ang and Chen, 2002 and Longin and Solnik, 2001). In the jargon of copula functions, this phenomenon is known as radial asymmetry. The effects of this kind of asymmetry on both asset management and pricing have to be explored further.

Non exchangeability. A problem with the use of many copula functions is that they are “exchangeable”. The term was first introduced in probability theory by De Finetti in the 30s. A bivariate distribution is deemed exchangeable if $C(u,v) = C(v,u)$. If we argue about the economic meaning of this concept, non-exchangeability points to a “dominant” role of a market or risk factor with respect to others. A somewhat close concept is that of “contagion”: something that happens to a market propagates to the others more than the other way around. Research in this field is at the very beginning.

Lévy processes. Recently, a more general approach to the dynamics of asset prices has gathered momentum and is regarded as the most promising and complete model of statistical finance for the future. The statistical processes used are called Lévy processes (see, for example, Carr et al. 2002 and Geman 2002). The idea is that the shocks affecting the price of an asset may be of three different kinds: diffusive processes, finite activity jumps and infinite activity jumps. Understanding correlation between prices gets then more involved, because one could model in a different way the dependence

structure between the diffusive and the jump components. A first contribution in this direction is the proposal of Lévy copulas (Kallsen and Tankov, 2006), but research in this field is still at the very beginning.

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